

Enveloping actions and duality theorems for partial twisted smash products

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Abstract In this paper, we first generalize the theorem about the existence of an enveloping action to a partial twisted smash product. Second, we construct a Morita context between the partial twisted smash product and the twisted smash product related to the enveloping action. Furthermore, we show some results relating partial actions and partial representations over the partial twisted smash products, which generalize the results of Alves and Batista (Comm. Algebra, 38(8): 2872-2902, 2010). Finally, we present versions of the duality theorems of Blattner-Montgomery for partial twisted smash products.

Key words Enveloping action, partial twisted smash product, Morita context, duality theorem.

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1 Introduction

Partial group actions were considered first by Exel in the context of operator algebras and they turned out to be a powerful tool in the study of C^* -algebras generated by partial isometries on a Hilbert space in [10]. A treatment from a purely algebraic point of view was given recently in [7], [8], [9]. In particular, the algebraic study of partial actions and partial representations was initiated in [8] and [9], motivating investigations in diverse directions. Now, the results are formulated in a purely algebraic way independent of the C^* -algebraic techniques which originated them.

The concepts of partial actions and partial coactions of Hopf algebras on algebras were introduced by Caenepeel and Janssen in [6]. In which they put the Galois theory for partial group actions on rings into a broader context, namely, the partial entwining structures. In

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particular, partial actions of a group G determine partial actions of the group algebra kG in a natural way. Further developments in the theory of partial Hopf actions were done by Lomp in [12].

Alves and Batista extended several results from the theory of partial group actions to the Hopf algebra setting, they constructed a Morita context relating the fixed point subalgebra for partial actions of finite dimensional Hopf algebras, and constructed the partial smash product in [1]. Later, they constructed a Morita context between the partial smash product and the smash product related to the enveloping action, defined partial representations of Hopf algebras and showed some results relating partial actions and partial representations in [2].

Furthermore, they proved a dual version of the globalization theorem: Every partial coaction of a Hopf algebra admits an enveloping coaction. they explored some consequences of globalization theorems in order to present versions of the duality theorems of Cohen-Montgomery and Blattner-Montgomery for partial Hopf actions in [3]. Recently, they introduced partial representations of Hopf algebras and gave the paradigmatic examples of them, namely, the partial representation defined from a partial action and the partial representation related to the partial smash product in [4].

Based on the existing results, this paper is organized as follows:

In Section 3, we introduce the notion of partial actions of a Hopf algebra containing a partial left action and a partial right action, and define a partial bimodule of a Hopf algebra. Then we introduce a new notion-partial twisted smash product $\underline{A \circledast H}$, generalizing the twisted smash product in [15]. Furthermore, we prove the existence of an enveloping action for such a partial twisted smash product.

In Section 4, we construct a Morita context between the partial twisted smash product $\underline{A \circledast H}$ and the twisted smash product $B \circledast H$, where H is a Hopf algebra which acts partially on the unital algebra A , B is an enveloping action for partial actions. This result can also be found in [2] for the context of partial group actions.

In Section 5, we show that, under some certain conditions on the algebra A , the partial twisted smash product $\underline{A \circledast H}$ carries a partial representation of H .

In Section 6, we explore some consequences of globalization theorems in order to present versions of the duality theorems of Blattner-Montgomery for partial twisted smash products.

2 Preliminaries

Throughout the paper, let k be a fixed field and all algebraic systems are supposed to be over k . Let M be a vector space over k and let id_M the usual identity map. For the comultiplication Δ in a coalgebra C with a counit ε_C , we use the Sweedler-Heyneman's

notation (see Sweedler [14]): $\Delta(c) = c_{(1)} \otimes c_{(2)}$, for any $c \in C$.

We first recall some basic results and propositions that we will need later from Alves and Batista [1],[2].

2.1. Partial left module algebra Let H be a Hopf algebra and A an algebra. A is said to be a partial left H -module algebra if there exists a k -linear map $\rightharpoonup = \{\rightharpoonup: H \otimes A \rightarrow A\}$ satisfying the following conditions:

$$\begin{aligned} h \rightharpoonup (ab) &= (h_{(1)} \rightharpoonup a)(h_{(2)} \rightharpoonup b), \\ 1_H \rightharpoonup a &= a, \\ h \rightharpoonup (g \rightharpoonup a) &= (h_{(1)} \rightharpoonup 1_A)(h_{(2)}g \rightharpoonup a), \end{aligned}$$

for all $h, g \in H$ and $a, b \in A$.

3 Enveloping actions

In the context of partial actions of Hopf algebras, it is proved that a partial action of a Hopf algebra on a unital algebra A admits an enveloping action (B, θ) if and only if each of the ideals $\theta(A) \supseteq B$ is a unital algebra in [2]. In this section, we mainly extend this famous result to partial twisted smash products.

Now, we give the definition of a partial right H -module algebra similar to [1] as follows:

Definition 3.1. Let H be a Hopf algebra and A an algebra, A is said to be a partial right H -module algebra if there exists a k -linear map $\leftharpoonup = \{\leftharpoonup: A \otimes H \rightarrow A\}$ satisfying the following conditions:

$$\begin{aligned} (ab) \leftharpoonup h &= (a \leftharpoonup h_{(1)})(b \leftharpoonup h_{(2)}), \\ a \leftharpoonup 1_H &= a, \\ (a \leftharpoonup g) \leftharpoonup h &= (1_A \leftharpoonup h_{(1)})(a \leftharpoonup gh_{(2)}), \end{aligned}$$

for all $h, g \in H$ and $a, b \in A$.

Definition 3.2. Let H be a Hopf algebra and A an algebra. A is called a partial H -bimodule algebra if the following conditions hold:

- (i) A is not only a partial left H -module algebra with the partial left module action \rightharpoonup but also a partial right H -module algebra with the partial right module action \leftharpoonup .
- (ii) These two partial module structure maps satisfy the compatibility condition, i.e., $(h \rightharpoonup a) \leftharpoonup g = h \rightharpoonup (a \leftharpoonup g)$ for all $a \in A$ and $h, g \in H$.

Let H be a Hopf algebra with an antipode S and A a partial H -bimodule algebra. We first propose a multiplication on the vector space $A \otimes H$:

$$(a \circledast h)(b \circledast g) = a(h_{(1)} \rightharpoonup b \leftharpoonup S(h_{(3)})) \circledast h_{(2)}g,$$

for all $a, c \in A$ and $g, h \in H$. It is obvious that the multiplication is associative. In order to make it to be an unital algebra, we project onto the

$$\underline{A \circledast H} = (A \otimes H)(1_A \otimes 1_H).$$

Then we can deduce directly the form and the properties of typical elements of this algebra

$$\underline{a \circledast h} = a(h_{(1)} \rightharpoonup 1_A \leftharpoonup S(h_{(3)})) \otimes h_{(2)},$$

and finally verify that the product among typical elements satisfy

$$(\underline{a \circledast h})(\underline{b \circledast g}) = \underline{a(h_{(1)} \rightharpoonup b \leftharpoonup S(h_{(3)})) \circledast h_{(2)}g}, \quad (3.1)$$

for all $h, g \in H$ and $a, b \in A$.

From the above definitions, we have

Proposition 3.3. *With the notations as above, $\underline{A \circledast H}$ is an associative algebra with a multiplication given by Eq.(3.1) and with the unit $\underline{1_A \circledast 1_H}$, and call it by a partial twisted smash product, where 1_A is the unit of A .*

Proof. Similar to [1]. □

Definition 3.4. *Let H and A be Hopf algebras. A skew pair is a triple (A, H, σ) endowed with a k -linear maps $\sigma : A \otimes H \rightarrow k$ such that the following conditions are satisfied.*

- (1) $\sigma(ab, h) = \sigma(a, h_{(1)})\sigma(b, h_{(2)}),$
- (2) $\sigma(a_{(1)}, h)\sigma(a_{(2)}, g) = \sigma(1_A, g_{(1)})\sigma(a, g_{(2)}h) = \sigma(1_A, h_{(1)})\sigma(a, gh_{(2)}),$
- (3) $\sigma(a, 1) = \varepsilon(a),$

for all $h, g \in H$ and $a, b \in A$.

Example 3.5. *Let H be a Hopf algebra with a bijective antipode S and A a Hopf algebra. Suppose that (A, H, σ) is a skew pair, then we can define two actions of H and A : for any $h \in H, b \in A$,*

$$\begin{aligned} h \rightharpoonup b &= b_{(2)}\sigma(b_{(1)}, h), \\ b \leftharpoonup h &= b_{(1)}\sigma(b_{(2)}, (S^{-1})^2(h)). \end{aligned}$$

It follows that

$$\begin{aligned} \underline{a \circledast h} &= a((h_{(1)} \rightharpoonup 1_A \leftharpoonup S(h_{(3)})) \otimes h_{(2)}) \\ &= \sigma(1_A, h_{(1)})a \otimes h_{(2)}\sigma(1_A, S^{-1}(h_{(3)})). \end{aligned}$$

It is not hard to verify that $(A, \rightharpoonup, \leftharpoonup)$ is a partial H -bimodule algebra and the multiplication of $\underline{A \otimes H}$ is

$$(\underline{a \otimes h})(\underline{b \otimes k}) = \sigma(b_{(1)}, (h_{(1)})) \underline{ab_{(2)} \otimes h_{(2)}k} \sigma(b_{(3)}, S^{-1}(h_{(3)})),$$

for all $h, k \in H$ and $a, b \in A$. Then $\underline{A \otimes_\sigma H}$ is a partial twisted smash product.

Example 3.6. As a k -algebra, the four dimensional Hopf algebra H_4 is generated by two symbols c and x which satisfy the relations $c^2 = 1$, $x^2 = 0$ and $xc + cx = 0$. The coalgebra structure on H_4 is determined by

$$\Delta(c) = c \otimes c, \quad \Delta(x) = x \otimes 1 + c \otimes x, \quad \varepsilon(c) = 1, \quad \varepsilon(x) = 0.$$

Consequently, H_4 has the basis 1 (identity), c , x , cx , we now consider the dual H_4^* of H_4 . We have $H_4 \cong H_4^*$ (as Hopf algebras) via

$$1 \mapsto 1^* + c^*, \quad c \mapsto 1^* - c^*, \quad x \mapsto x^* + (cx)^*, \quad cx \mapsto x^* - (cx)^*.$$

Here $\{1^*, c^*, x^*, (cx)^*\}$ denote the dual basis of $\{1, c, x, cx\}$. Let $T = 1^* - c^*$, $P = x^* + (cx)^*$, $TP = x^* - (cx)^*$, we get another basis $\{1, T, P, TP\}$ of H_4^* . Recall from [6] if A is the subalgebra $k[x]$ of H_4 , it is shown that A is a right partial H_4 -comodule algebra with the coaction

$$\rho(1) = \frac{1}{2}(1 \otimes 1 + 1 \otimes c + 1 \otimes cx), \quad \rho^r(x) = \frac{1}{2}(x \otimes 1 + x \otimes c + x \otimes cx).$$

By similar way we can show that A is a left partial H_4 -comodule algebra with the coaction

$$\rho(1) = \frac{1}{2}(1 \otimes 1 + c \otimes 1 + cx \otimes 1), \quad \rho^l(x) = \frac{1}{2}(1 \otimes x + c \otimes x + cx \otimes x).$$

It is clear that A is a partial H_4 -bicomodule algebra. Then A is a partial H_4^* -bimodule algebra via

$$f \rightharpoonup a = \sum \langle f, a_{[1]} \rangle a_{[0]}, \quad a \leftharpoonup g = \langle g, a_{[-1]} \rangle a_{[0]}, \quad a \in A, f, g \in H^*.$$

Therefore we can obtain the partial twisted smash product $k[x] \otimes H_4^*$, we only consider the elements P, T of H_4^* as follows, then

$$\begin{aligned} (x \otimes T)(x \otimes P) &= x(T_{(1)} \rightharpoonup x \leftharpoonup S^*(T_{(3)})) \otimes T_{(2)}P \\ &= \sum x(T \rightharpoonup x \leftharpoonup S^*(T)) \otimes TP \\ &= \sum x \langle T, \frac{1}{2}(1 + c + cx) \rangle x \langle T, \frac{1}{2}(1 + c + cx) \rangle \otimes TP = 0. \end{aligned}$$

Definition 3.7. Let H be a Hopf algebra and A, B be two partial H -bimodule algebras. A morphism of algebras $\theta : A \rightarrow B$ is said to be a morphism of partial H -bimodule algebras if $\theta(h \rightharpoonup a \leftharpoonup k) = h \rightharpoonup \theta(a) \leftharpoonup k$ for all $h, k \in H$ and $a \in A$. If, in addition, θ is an isomorphism, the partial actions are called equivalent.

Lemma 3.8. *Let H be a Hopf algebra, B a H -bimodule algebra and A an ideal of B with unity 1_A . Then H acts partially on A by $h \rightharpoonup a = 1_A(h \triangleright a)$, $a \leftharpoonup h = (a \triangleleft h)1_A$, for all $a \in A, b \in B$ and $h \in H$.*

Proof. Similarly to [2]. □

Lemma 3.9. *Let H be a Hopf algebra and A an algebra. Then $(A, \rightharpoonup, \leftharpoonup)$ is a partial H -bimodule algebra.*

Proof. Similarly to [2]. □

Recall from [2] that if B is an H -module algebra and A is a right ideal of B with unity 1_A , the induced partial action on A is called admissible if $B = H \triangleright A$.

Similarly, if B is an H -module algebra and A is a left ideal of B with unity 1_A , the induced partial action on A is called admissible if $B = A \triangleleft H$.

Now, let $B = H \triangleright A$ and $B = A \triangleleft H$ at the same time, note it by $B = (A, \triangleright, \triangleleft)$.

Definition 3.10. *Let H be a Hopf algebra, B an H -bimodule algebra and A an ideal of B with unit 1_A . The induced partial actions on A is called admissible if $B = (A, \triangleright, \triangleleft)$.*

Definition 3.11. *Let A be a partial H -bimodule algebra. An enveloping action for A is a pair (B, θ) , where*

- (a) B is an H -bimodule algebra;
- (b) The map $\theta : A \rightarrow B$ is a monomorphism of algebras;
- (c) The sub-algebra $\theta(A)$ is an ideal in B ;
- (d) The partial action on A is equivalent to the induced partial action on $\theta(A)$;
- (e) The induced partial action on $\theta(A)$ is admissible.

From now on, we always assume that $(a \leftharpoonup S(h_{(1)})) \otimes h_{(2)} = (a \leftharpoonup S(h_{(2)})) \otimes h_{(1)}$ for any $a \in A$ and $h \in H$. This condition can be easily verified for the case where H^* is cocommutative (therefore, H is commutative). It is reasonable to assume this condition beyond the case of H being a cocommutative Hopf algebra.

A concrete counterexample is presented as follows. Recall from [6] that if $e \in H$ is a central idempotent such that $e \otimes e = (e \otimes 1)\Delta(e) = \Delta(e)(e \otimes 1)$ and $\varepsilon(e) = 1$. A partial H -coaction on $A = k$ is given by $\rho^l(x) = e \otimes x \in H \otimes_k k$ and $\rho^r(x) = x \otimes e \in k \otimes_k H$. It is easy to check that A is a partial H -bicomodule algebra. Therefore, A is a partial H^* -bimodule algebra. So we can obtain the partial twisted smash product $k \otimes H^*$, here we only consider the element f of H^* and check the condition. Then we get $(a \leftharpoonup S^*(h_{(2)})) \otimes h_{(1)} = (x \leftharpoonup S^*(f_{(2)})) \otimes f_{(1)} = x \otimes \langle f_{(2)}, e \rangle f_{(1)}$ and

$$\begin{aligned} (a \leftharpoonup S^*(h_{(1)})) \otimes h_{(2)} &= (x \leftharpoonup S^*(f_{(1)})) \otimes f_{(2)} \\ &= \langle f_{(1)}, e \rangle x \otimes f_{(2)} = x \otimes \langle f_{(1)}, e \rangle f_{(2)}. \end{aligned}$$

We can easily obtain $\langle f_{(1)}, e \rangle f_{(2)} = \langle f_{(2)}, e \rangle f_{(1)}$ since $e \in H$ is a central idempotent.

Lemma 3.12. Let $\varphi : A \rightarrow \text{Hom}(H, A)$ be the map given by $\varphi(a)(h) = h_{(1)} \rightharpoonup a \leftharpoonup S(h_{(2)})$, then we have

- (i) φ is a linear injective map and an algebra morphism;
- (ii) $\varphi(1_A) * (h_{(1)} \triangleright \varphi(a) \triangleleft S(h_{(2)})) = \varphi(h_1 \rightharpoonup a \leftharpoonup S(h_{(2)}))$ for any $h \in H$ and $a \in A$;
- (iii) $\varphi(b) * (h_1 \triangleright \varphi(a) \triangleleft S(h_{(2)})) = \varphi(b(h_{(1)} \rightharpoonup a \leftharpoonup S(h_{(2)})))$ for any $h \in H$ and $a \in A$.

Proof. It is easy to see that φ is linear, because the partial action is bilinear. Since $\varphi(a)(1_H) = a$, it follows that it is also injective. For any $a, b \in A$ and $h \in H$, we have

$$\begin{aligned}
\varphi(ab)(h) &= h_{(1)} \rightharpoonup (ab) \leftharpoonup S(h_{(2)}) \\
&= [h_{(1)} \rightharpoonup (a) \leftharpoonup S(h_{(4)})][h_{(2)} \rightharpoonup (b) \leftharpoonup S(h_{(3)})] \\
&= [h_{(1)} \rightharpoonup (a) \leftharpoonup S(h_{(2)})][h_{(3)} \rightharpoonup (b) \leftharpoonup S(h_{(4)})] \\
&= \varphi(a)(h_{(1)})\varphi(b)(h_{(2)}) = \varphi(a) * \varphi(b)(h).
\end{aligned}$$

Therefore φ is multiplicative.

For the third claim, we have the following calculation:

$$\begin{aligned}
&\varphi(b(h_{(1)} \rightharpoonup a \leftharpoonup S(h_{(2)})))(k) \\
&= k_{(1)} \rightharpoonup b(h_{(1)} \rightharpoonup a \leftharpoonup S(h_{(2)})) \leftharpoonup S(k_{(2)}) \\
&= [k_{(1)} \rightharpoonup b \leftharpoonup S(k_{(4)})][k_{(2)} \rightharpoonup (h_{(1)} \rightharpoonup a \leftharpoonup S(h_{(2)})) \leftharpoonup S(k_{(3)})] \\
&= [k_{(1)} \rightharpoonup b \leftharpoonup S(k_{(6)})][k_{(2)} \rightharpoonup 1_A \leftharpoonup S(k_{(5)})][k_{(3)}h_{(1)} \rightharpoonup a \leftharpoonup S(k_{(4)}h_{(2)})] \\
&= [k_{(1)} \rightharpoonup b \leftharpoonup S(k_{(4)})][k_{(2)}h_{(1)} \rightharpoonup a \leftharpoonup S(k_{(3)}h_{(2)})] \\
&= [k_{(1)} \rightharpoonup b \leftharpoonup S(k_{(2)})][k_{(3)}h_{(1)} \rightharpoonup a \leftharpoonup S(k_{(4)}h_{(2)})] \\
&= \varphi(b)(k_1)\varphi(a)(k_2)h \\
&= \varphi(b)(k_1)(h \triangleright \varphi(a))(k_2) = \varphi(b) * (h \triangleright \varphi(a))(k).
\end{aligned}$$

Therefore, $\varphi(h \rightharpoonup a) = \varphi(1_A)(h \triangleright \varphi(a))$. One may obtain the second item by setting $b = 1_A$. \square

Proposition 3.13. Let $\varphi : A \rightarrow \text{Hom}(H, A)$ be the map defined in Lemma 3.12 and $B = (\varphi(A), \triangleright, \triangleleft)$ the H -submodule of $\varphi(A)$. Then

- (i) B is an H -module subalgebra of $\text{Hom}(H, A)$;
- (ii) $\varphi(A)$ is a right ideal in B with unity $\varphi(1_A)$.

Proof. Similar to [2]. \square

By Lemma 3.12 and Proposition 3.13, we obtain the main result of this section.

Theorem 3.14. Let A be a partial H -bimodule algebra and $\varphi : A \rightarrow \text{Hom}(H, A)$ the map given by $\varphi(a)(h) = h_{(1)} \rightharpoonup a \leftharpoonup S(h_{(2)})$. Assume that $B = (\varphi(A), \triangleright, \triangleleft)$, then (B, φ) is an enveloping action of A .

Proposition 3.15. *Let A be a partial H -bimodule algebra and $\varphi : A \rightarrow \text{Hom}(H, A)$ the map given by $\varphi(a)(h) = h_{(1)} \rightharpoonup a \leftarrow S(h_{(2)})$. Assume that $B = (\varphi(A), \triangleright, \triangleleft)$, then $\varphi(A) \supseteq B$ if and only if*

$$k_{(1)} \rightharpoonup (h \rightharpoonup a) \leftarrow S(k_{(2)}) = [k_{(1)}h_{(1)} \rightharpoonup a \leftarrow S(k_{(2)}h_{(2)})][k_{(3)} \rightharpoonup 1_A \leftarrow S(k_{(4)})].$$

Proof. Suppose that $\varphi(A)$ is an ideal of B . We know that

$$\varphi(h \rightharpoonup a) = \varphi(1_A) * (h \triangleright \varphi(a)) = (h \triangleright \varphi(a)) * \varphi(1_A).$$

Then, these two functions coincide for all $k \in H$,

$$\varphi(h \rightharpoonup a)(k) = (h \triangleright \varphi(a)) * \varphi(1_A)(k).$$

The left-hand side of the previous equality leads to

$$\varphi(h \rightharpoonup a)(k) = k_{(1)} \rightharpoonup (h \rightharpoonup a) \leftarrow S(k_{(2)}).$$

While the right-hand side means

$$\begin{aligned} (h \triangleright \varphi(a)) * \varphi(1_A)(k) &= (h \triangleright \varphi(a))(k_{(1)})\varphi(1_A)(k_{(2)}) \\ &= \varphi(a)(k_{(1)}h)\varphi(1_A)(k_{(2)}) \\ &= [k_{(1)}h_{(1)} \rightharpoonup a \leftarrow S(k_{(2)}h_{(2)})][k_{(3)} \rightharpoonup 1_A \leftarrow S(k_{(4)})]. \end{aligned}$$

Conversely, suppose that the equality

$$k_{(1)} \rightharpoonup (h \rightharpoonup a) \leftarrow S(k_{(2)}) = [k_{(1)}h_{(1)} \rightharpoonup a \leftarrow S(k_{(2)}h_{(2)})][k_{(3)} \rightharpoonup 1_A \leftarrow S(k_{(4)})]$$

holds for all $a \in A$ and $h, k \in H$. Then $\varphi(1_A)$ is a central idempotent in B . Therefore $\varphi(A) = \varphi(1_A)B$ is an ideal in B . \square

4 A Morita context

In this section, we will construct a Morita context between the partial twisted smash product $\underline{A} \circledast \underline{H}$ and the twisted smash product $B \circledast H$, where B is an enveloping action for the partial twisted smash product.

Lemma 4.1. *Let A be a partial H -bimodule algebra and (B, θ) an enveloping action, then there is an algebra monomorphism from the partial twisted smash product $\underline{A} \circledast \underline{H}$ into the twisted smash product $B \circledast H$.*

Proof. Define $\Phi : A \otimes H \rightarrow B \otimes H$ by $a \otimes h \mapsto \theta(a) \otimes h$ for $h, g \in H$ and $a, b \in A$. We first check that Φ is a morphism of algebras as follows:

$$\begin{aligned}
\Phi((a \otimes h)(b \otimes g)) &= \Phi(a(h_{(1)} \rightharpoonup b \leftharpoonup S(h_{(3)})) \otimes h_{(2)}g) \\
&= \theta(a(h_{(1)} \rightharpoonup b \leftharpoonup S(h_{(3)})) \otimes h_{(2)}g) \\
&= \theta(a)(h_{(1)} \rhd \theta(b) \triangleleft S(h_{(3)})) \otimes h_{(2)}g \\
&= (\theta(a) \otimes h)(\theta(b) \otimes g) \\
&= \Phi(a \otimes h)\Phi(b \otimes g).
\end{aligned}$$

Next, we will verify that Φ is injective. For this purpose, take $x = \sum_{i=1}^n a_i \otimes h_i \in \ker \Phi$ and choose $\{a_i\}_{i=1}^n$ to be linearly independent. Since θ is injective, we conclude that $\theta(a_i)$ are linearly independent. For each $f \in H^*$, $\sum_{i=1}^n \theta(a_i)f(h_i) = 0$, it follows that $f(h_i) = 0$, so $h_i = 0$. Therefore we have $x = 0$ and Φ is injective, as desired.

Since the partial twisted smash product $\underline{A \otimes H}$ is a subalgebra of $A \otimes H$, it is injectively mapped into $B \otimes H$ by Φ . A typical element of the image of the partial twisted smash product is

$$\begin{aligned}
\Phi((a \otimes h)(1_A \otimes 1_H)) &= \Phi(a \otimes h)\Phi(1_A \otimes 1_H) \\
&= (\theta(a) \otimes h)(\theta(1_A) \otimes 1_H) \\
&= \theta(a(h_{(1)} \rhd \theta(1_A) \triangleleft S(h_{(3)}))) \otimes h_{(2)}g.
\end{aligned}$$

And this completes the proof. \square

Take $M = \Phi(A \otimes H) = \{\sum_{i=1}^n \theta(a_i) \otimes h_i; a_i \in A\}$ and take N as the subspace of $B \otimes H$ generated by the elements $(h_{(1)} \rhd \theta(a) \triangleleft S(h_{(3)})) \otimes h_{(2)}$ with $h \in H$ and $a \in A$.

Proposition 4.2. *Let H be a Hopf algebra with an invertible antipode S and A a partial H -bimodule algebra. Suppose that $\theta(A)$ is an ideal of B , then M is a right $B \otimes H$ module and N is a left $B \otimes H$ module.*

Proof. In order to prove M is a right $B \otimes H$ module, let $\theta(a) \otimes h \in M$ and $b \otimes k \in B \otimes H$, then

$$(\theta(a) \otimes h)(b \otimes k) = \theta(a)(h_{(1)} \rhd b \triangleleft S(h_{(3)})) \otimes h_{(2)}k.$$

Which lies in $\Phi(A \otimes H)$ because $\theta(A)$ is an ideal in B .

Now we show that N is a left $B \otimes H$ module. Let $(h_{(1)} \rhd \theta(a) \triangleleft S(h_{(3)})) \otimes h_{(2)}$, where

$h \in H$ is a generator of N , then we have

$$\begin{aligned}
& (b \otimes k)(h_{(1)} \triangleright \theta(a) \triangleleft S(h_{(3)})) \otimes h_{(2)} \\
&= b(k_{(1)}h_{(1)}) \triangleright \theta(a) \triangleleft S(k_{(3)}h_{(3)}) \otimes k_{(2)}h_{(2)} \\
&= [(\varepsilon(k_{(1)}h_{(1)}) \triangleright b)(k_{(2)}h_{(2)}) \triangleright \theta(a)] \triangleleft S(k_{(4)}h_{(4)}) \otimes k_{(3)}h_{(3)} \\
&= [((k_{(2)}h_{(2)})S(k_{(1)}h_{(1)}) \triangleright b(k_{(3)}h_{(3)}) \triangleright \theta(a)) \triangleleft S(k_{(5)}h_{(5)})] \otimes k_{(4)}h_{(4)} \\
&= [(k_{(2)}h_{(2)}) \triangleright ((S(k_{(1)}h_{(1)}) \triangleright b)(k_{(2)}h_{(2)}) \triangleright \theta(a)) \triangleleft S(k_{(4)}h_{(4)})] \otimes k_{(3)}h_{(3)} \\
&= [(k_{(2)}h_{(2)}) \triangleright (S(k_{(1)}h_{(1)}) \triangleright b)\theta(a) \triangleleft S(k_{(4)}h_{(4)})] \otimes k_{(3)}h_{(3)}.
\end{aligned}$$

Because $\theta(A)$ is an ideal of B , it follows that N is a left $B \otimes H$ module. \square

By Proposition 4.2, we can define a left $\underline{A \otimes H}$ module structure on M and a right $\underline{A \otimes H}$ module structure on N induced by the monomorphism Φ as follows:

$$\begin{aligned}
& (ah_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}) \otimes h_{(2)}) \blacktriangleright (\theta(b)) \otimes k \\
&= (\theta(a)h_{(1)} \triangleright \theta(1_A) \triangleleft S(h_{(3)}) \otimes h_{(2)})(\theta(b)) \otimes k, \\
& ((k_{(1)}) \triangleright \theta(b) \triangleleft S(k_{(3)})) \otimes k_{(2)} \blacktriangleleft (ah_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}) \otimes h_{(2)}) \\
&= ((k_{(1)}) \triangleright \theta(b) \triangleleft S(k_{(3)})) \otimes k_{(2)} (\theta(a)h_{(1)} \triangleright \theta(1_A) \triangleleft S(h_{(3)}) \otimes h_{(2)}).
\end{aligned}$$

Proposition 4.3. *Under the same hypotheses of Proposition 4.2, M is indeed a left $\underline{A \otimes H}$ module with the map \blacktriangleright and N is a right $\underline{A \otimes H}$ module with the map \blacktriangleleft .*

Proof. We first claim that $\underline{A \otimes H} \blacktriangleright M \subseteq M$. In fact,

$$\begin{aligned}
& (ah_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}) \otimes h_{(2)}) \blacktriangleright (\theta(b)) \otimes k \\
&= (\theta(a)(h_{(1)} \triangleright \theta(1_A) \triangleleft S(h_{(3)}) \otimes h_{(2)})(\theta(b)) \otimes k) \\
&= \theta(a)(h_{(1)} \triangleright \theta(1_A) \triangleleft S(h_{(5)}))(h_{(2)}) \rightharpoonup \theta(b) \leftarrow S(h_{(4)}) \otimes h_{(3)}k_{(2)} \\
&= \theta(a)((h_{(1)}) \triangleright \theta(b) \triangleleft S(h_{(3)})) \otimes h_{(2)}k_{(2)}.
\end{aligned}$$

Which lies inside M because $\theta(A)$ is an ideal of B . \square

Next, we verify that $N \blacktriangleleft \underline{A \otimes H} \subseteq N$, which is similarly to N is a left $B \otimes H$ module. which holds because $\theta(1_A)$ is a central idempotent.

The last ingredient for a Morita context is to define two bimodule morphisms

$$\begin{aligned}
& \sigma : N \otimes_{\underline{A \otimes H}} M \rightarrow B \otimes H \quad \text{and} \\
& \tau : M \otimes_{B \otimes H} N \rightarrow \underline{A \otimes H} \cong \Phi(\underline{A \otimes H}).
\end{aligned}$$

As M, N and $\underline{A \otimes H}$ are viewed as subalgebras of $B \otimes H$, these two maps can be taken as the usual multiplication on $B \otimes H$. The associativity of the product assures us that these maps are bimodule morphisms and satisfy the associativity conditions.

Proposition 4.4. *The partial twisted smash product $\underline{A \otimes H}$ is Morita equivalent to the twisted smash product $B \otimes H$.*

5 Partial representation

In [2], Alves and Batista introduced the notion of a partial representation of a Hopf algebra. In this section, we will show a partial representation of the partial twisted smash product $A \otimes H$.

Proposition 5.1. *Let H be a Hopf algebra with an invertible antipode and A a partial H -bimodule algebra. Then the map*

$$\pi : H \rightarrow \text{End}(A), \quad h \mapsto \pi(h)$$

given by $\pi(h)(a) = h_{(1)} \rightharpoonup a \leftarrow S(h_{(2)})$ satisfies:

- (1) $\pi(1_H) = 1_A$;
- (2) $\pi(S^{-1}(h_{(2)}))\pi(h_{(1)})\pi(k) = \pi(S^{-1}(h_{(2)}))\pi(h_{(1)}k)$.

Proof. The first identity is quite obvious. In order to prove the second equality, we take $a \in A$ and do the following calculation:

$$\begin{aligned} & \pi(S^{-1}(h_{(2)}))\pi(h_{(1)})\pi(k)(a) \\ &= S^{-1}(h_{(4)}) \rightharpoonup [h_{(1)} \rightharpoonup (k_{(1)} \rightharpoonup a \leftarrow S(k_{(2)})) \leftarrow S(h_{(2)})] \leftarrow h_{(3)} \\ &= (S^{-1}(h_{(6)}) \rightharpoonup 1_A \leftarrow h_{(3)})[S^{-1}(h_{(5)})h_{(1)} \rightharpoonup (k_{(1)} \rightharpoonup a \leftarrow S(k_{(2)})) \leftarrow S(h_{(2)})h_{(4)}] \\ &= (S^{-1}(h_{(6)}) \rightharpoonup 1_A \leftarrow h_{(5)})[S^{-1}(h_{(2)})h_{(1)} \rightharpoonup (k_{(1)} \rightharpoonup a \leftarrow S(k_{(2)})) \leftarrow S(h_{(3)})h_{(4)}] \\ &= (S^{-1}(h_{(2)}) \rightharpoonup 1_A \leftarrow h_{(1)})(k_{(1)} \rightharpoonup a \leftarrow S(k_{(2)})). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \pi(S^{-1}(h_{(2)}))\pi(h_{(1)}k)(a) \\ &= S^{-1}(h_{(4)}) \rightharpoonup [h_{(1)}k_{(1)} \rightharpoonup a \leftarrow S(h_{(2)}k_{(2)})] \leftarrow h_{(3)} \\ &= (S^{-1}(h_{(6)}) \rightharpoonup 1_A \leftarrow h_{(3)})[S^{-1}(h_{(5)})h_{(1)}k_{(1)} \rightharpoonup a \leftarrow S(h_{(2)}k_{(2)})h_{(4)}] \\ &= (S^{-1}(h_{(6)}) \rightharpoonup 1_A \leftarrow h_{(5)})[S^{-1}(h_{(2)})h_{(1)}k_{(1)} \rightharpoonup a \leftarrow S(h_{(3)}k_{(2)})h_{(4)}] \\ &= (S^{-1}(h_{(2)}) \rightharpoonup 1_A \leftarrow h_{(1)})(k_{(1)} \rightharpoonup a \leftarrow S(k_{(2)})). \end{aligned}$$

And this completes the proof. □

With this result we can propose the following definition similar to [2].

Definition 5.2. *Let H be a Hopf algebra with an invertible antipode. A partial representation of H on a unital algebra B is a linear map*

$$\pi : H \rightarrow B, \quad h \mapsto \pi(h)$$

such that

- (1) $\pi(1_H) = 1_B$;
- (2) $\pi(S^{-1}(h_{(2)}))\pi(h_{(1)})\pi(k) = \pi(S^{-1}(h_{(2)}))\pi(h_{(1)}k)$.

Theorem 5.3. *Let H be a Hopf algebra with an invertible antipode and A a partial H -bimodule algebra. Then the linear map*

$$\pi : H \rightarrow \underline{A \otimes H}, \quad h \mapsto (h_{(1)} \rightharpoonup 1_A \leftarrow S(h_3)) \otimes h_{(2)}$$

is a partial representation of H .

Proof. The first identity is quite obvious. In order to prove the second equality, we need the following calculations:

$$\begin{aligned} & \pi(S^{-1}(h_{(2)}))\pi(h_{(1)})\pi(k) \\ &= ((S^{-1}(h_{(6)}) \rightharpoonup 1_A \leftarrow h_{(4)}) \otimes S^{-1}(h_{(5)}))(h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}) \otimes h_{(2)}) \\ & \quad (k_{(1)} \rightharpoonup 1_A \leftarrow S(k_{(3)}) \otimes k_{(2)}) \\ &= (S^{-1}(h_{(8)}) \rightharpoonup 1_A \leftarrow h_{(4)})[S^{-1}(h_{(7)}) \rightharpoonup (h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}) \leftarrow h_{(5)})] \\ & \quad \otimes S^{-1}(h_{(6)})h_{(2)})(k_{(1)} \rightharpoonup 1_A \leftarrow S(k_{(3)}) \otimes k_{(2)}) \\ &= (S^{-1}(h_{(10)}) \rightharpoonup 1_A \leftarrow h_{(4)})(S^{-1}(h_{(9)}) \rightharpoonup 1_A \leftarrow h_{(5)}) \\ & \quad [(S^{-1}(h_{(8)})h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)})h_{(6)}) \otimes S^{-1}(h_{(7)})h_{(2)}](k_{(1)} \rightharpoonup 1_A \leftarrow S(k_{(3)}) \otimes k_{(2)}) \\ &= (S^{-1}(h_{(8)}) \rightharpoonup 1_A \leftarrow h_{(4)})[(S^{-1}(h_{(7)})h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)})h_{(5)})] \\ & \quad \otimes S^{-1}(h_{(6)})h_{(2)})(k_{(1)} \rightharpoonup 1_A \leftarrow S(k_{(3)}) \otimes k_{(2)}) \\ &= (S^{-1}(h_{(7)}) \rightharpoonup 1_A \leftarrow h_{(2)})[(S^{-1}(h_{(6)})h_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)})h_{(4)})] \\ & \quad \otimes S^{-1}(h_{(6)})h_{(5)})(k_{(1)} \rightharpoonup 1_A \leftarrow S(k_{(3)}) \otimes k_{(2)}) \\ &= (S^{-1}(h_{(2)}) \rightharpoonup 1_A \leftarrow h_{(1)})(k_{(1)} \rightharpoonup 1_A \leftarrow S(k_{(3)}) \otimes k_{(2)}). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \pi(S^{-1}(h_{(2)}))\pi(h_{(1)})\pi(k) \\ &= [S^{-1}(h_{(6)}) \rightharpoonup 1_A \leftarrow h_{(4)} \otimes S^{-1}(h_{(5)})][h_{(1)}k_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}k_{(3)}) \otimes h_{(2)}k_{(2)}] \\ &= (S^{-1}(h_{(8)}) \rightharpoonup 1_A \leftarrow h_{(4)})S^{-1}(h_{(7)}) \rightharpoonup [h_{(1)}k_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}k_{(3)})] \leftarrow h_{(5)} \\ & \quad \otimes S^{-1}(h_{(6)})h_{(2)}k_{(2)}] \\ &= (S^{-1}(h_{(10)}) \rightharpoonup 1_A \leftarrow h_{(4)})(S^{-1}(h_{(9)}) \rightharpoonup 1_A \leftarrow h_{(5)}) \\ & \quad [S^{-1}(h_{(8)})h_{(1)}k_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}k_{(3)})h_{(6)}] \otimes S^{-1}(h_{(7)})h_{(2)}k_{(2)}] \\ &= (S^{-1}(h_{(8)}) \rightharpoonup 1_A \leftarrow h_{(4)})[S^{-1}(h_{(7)})h_{(1)}k_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}k_{(3)})h_{(5)}] \\ & \quad \otimes S^{-1}(h_{(6)})h_{(2)}k_{(2)}] \\ &= (S^{-1}(h_{(8)}) \rightharpoonup 1_A \leftarrow h_{(7)})[S^{-1}(h_{(2)})h_{(1)}k_{(1)} \rightharpoonup 1_A \leftarrow S(h_{(3)}k_{(3)})h_{(4)}] \\ & \quad \otimes S^{-1}(h_{(6)})h_{(5)}k_{(2)}] \\ &= (S^{-1}(h_{(2)}) \rightharpoonup 1_A \leftarrow h_{(1)})(k_{(1)} \rightharpoonup 1_A \leftarrow S(k_{(3)}) \otimes k_{(2)}). \end{aligned}$$

So $\pi(S^{-1}(h_{(2)}))\pi(h_{(1)})\pi(k) = \pi(S^{-1}(h_{(2)}))\pi(h_{(1)}k)$, as required. \square

6 Duality for partial twisted smash products

In this section, we explore some consequences of globalization theorems in order to present versions of the duality theorems of Blattner-Montgomery for partial twisted smash products, generalizing the results of [12].

Let H be a Hopf algebra which is finitely generated and projective as k -module with dual basis $\{(b_i, p_i) \in H \otimes H^* | 1 \leq i \leq n\}$. Assume that H^* acts on H from the left by $f \rightarrow h = \sum h_{(1)}f(h_{(2)})$ and the right by $h \leftarrow f = \sum h_{(2)}f(h_{(1)})$, such that the smash product $H \# H^*$ can be considered as an algebra whose multiplication is given by

$$(h \# f)(k \# g) = \sum h(f_{(1)} \rightarrow k) \# f_{(2)} * g,$$

for any $h, k \in H$ and $f, g \in H^*$.

Lemma 6.1. [14] *Let H be a finite dimensional Hopf algebra. Then the linear maps*

$$(1) \lambda : H \# H^* \rightarrow \text{End}(H), \quad \lambda(h \# f)(k) = h(f \rightarrow k),$$

$$(2) \varphi : H^* \# H \rightarrow \text{End}(H), \quad \varphi(f \# h)(k) = (k \leftarrow f)h,$$

are isomorphisms of algebras, where $h, k \in H$ and $f, g \in H^$.*

The partial twisted smash product $\underline{A \otimes H}$ in Proposition 3.3. becomes naturally a right H -comodule algebra by

$$\rho = 1 \otimes \Delta : A \otimes H \otimes H \rightarrow A \otimes H \otimes H, \quad a \otimes h \mapsto a \otimes h_{(1)} \otimes h_{(2)}.$$

For $(a \otimes h)1_A \in \underline{A \otimes H}$, we have

$$\rho((a \otimes h)1_A) = a(h_{(1)} \rightarrow 1_A \leftarrow S(h_{(2)})) \otimes h_{(3)} \otimes h_{(4)},$$

which make $\underline{A \otimes H}$ into a right H -comodule algebra. Moreover, $\underline{A \otimes H}$ becomes a left H^* module algebra, where the action is defined by

$$f \cdot ((a \otimes h)1_A) = a(h_{(1)} \rightarrow 1_A \leftarrow S(h_{(3)})) \# (f \rightarrow h_{(2)}) = (a \# (f \rightarrow h))1_A.$$

for all $f \in H^*, h \in H, a \in A$.

Similar to [12], we can define a homomorphism $\phi : A \rightarrow A \otimes \text{End}(H)$ by

$$\phi(a) = \sum_{i=1}^n (b_{i(1)} \rightarrow a \leftarrow S(b_{i(2)})) \otimes \varphi(S^{-1}(p_i) \otimes 1_H).$$

Then ϕ is an algebra homomorphism.

Lemma 6.2. *Let $\psi : H \# H^* \rightarrow A \otimes \text{End}(H)$ be the map defined by $h \# f \mapsto 1 \otimes \lambda(h \# f)$ for all $h \in H$ and $f \in H^*$. Then we have*

$$\phi(1_A)\psi(h \# f)\phi(a) = \phi(h_{(1)} \rightarrow a \leftarrow S(h_{(3)}))\psi(h_{(2)} \# f).$$

Proof. For any $h \in H, f \in H^*$ and $a \in A$, we have

$$\begin{aligned}
& \phi(h_{(1)} \rightarrow a \leftarrow S(h_{(3)}))\psi(h_{(2)}\#f) \\
&= \sum_i p_i(h_{(1)})\phi(b_{i(1)} \rightarrow a \leftarrow S(b_{i(3)}))\psi(h_{(2)}\#f) \\
&= \sum_{i,j} b_{j(1)} \rightarrow (b_{i(1)} \rightarrow a \leftarrow S(b_{i(3)})) \leftarrow S(b_{j(3)}) \otimes \varphi(S^{-1}(p_j)\#1_H)\lambda(h \leftarrow p_i\#f) \\
&= \sum_{k,r} (b_{k(1)} \rightarrow 1_A \leftarrow S(b_{k(2)}))(b_{r(1)} \rightarrow a \leftarrow S(b_{r(2)})) \otimes \varphi(S^{-1}(p_k)\#1_H) \\
&\quad \varphi(S^{-1}(p_{k(1)})\#1_H)\lambda(h \leftarrow p_{k(2)}\#f) \\
&= \phi(1_A) \sum_r (b_{r(1)} \rightarrow a \leftarrow S(b_{r(2)})) \otimes \varphi(S^{-1}(p_r)_{(2)}\#1_H)\lambda(h \leftarrow S(S^{-1}(p_r)_{(1)})\#f) \\
&= \phi(1_A) \sum_r (b_{r(1)} \rightarrow a \leftarrow S(b_{r(2)})) \otimes \lambda(h\#f)\varphi(S^{-1}(p_r)\#1_H) \\
&= \phi(1_A)\psi(h\#f)\phi(a).
\end{aligned}$$

The proof is completed. \square

Theorem 6.3. *Let H be a finitely generated projective Hopf algebra and A a partial H -bimodule algebra. Then the map*

$$\Phi : A \otimes H\#H^* \rightarrow A \otimes \text{End}(H), \quad a \otimes h\#f \mapsto \phi(a)\psi(h\#f)$$

is an algebra homomorphism. The image of the restriction to $\underline{A \otimes H}\#H^$ lies inside $e(A \otimes \text{End}(H))e$, where e is the idempotent defined by*

$$e = \sum_{i=1}^n (b_{i(1)} \rightarrow 1_A \leftarrow S(b_{i(2)})) \otimes \varphi(S^{-1}(p_i) \otimes 1_A).$$

Proof. For any $a, b \in A, h, k \in H$ and $f, g \in H^*$, we have

$$\begin{aligned}
\Phi(a \otimes h\#f)\Phi(b \otimes k\#g) &= \phi(a)\psi(h\#f)\phi(b)\psi(k\#g) \\
&= \phi(a)\phi(1_A)\psi(h\#f)\phi(b)\psi(k\#g) \\
&= \phi(a)\phi(h_{(1)} \rightarrow b \leftarrow S(h_{(3)}))\psi(h_{(2)}\#f)\psi(k\#g) \\
&= \phi(a(h_{(1)} \rightarrow b \leftarrow S(h_{(3)})))\psi(h_{(2)}(f_{(1)} \rightarrow k)\#f_{(2)} * g) \\
&= \Phi(\phi(a(h_{(1)} \rightarrow b \leftarrow S(h_{(3)}))) \otimes h_{(2)}(f_{(1)} \rightarrow k)\#f_{(2)} * g) \\
&= \Phi((a \otimes h\#f)(b \otimes k\#g)).
\end{aligned}$$

Hence Φ is an algebra homomorphism. Since the image of the identity $1 = 1_A \otimes 1_H\#1_{H^*}$ of $\underline{A \otimes H}\#H^*$ under the map Φ is e , e is an idempotent. Moreover, for any $\gamma \in \underline{A \otimes H}\#H^*$, we have

$$\Phi(\gamma) = \Phi(1_A \gamma 1_A) \in e(A \otimes \text{End}(H))e,$$

as desired. And this completes the proof. \square

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